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Quantization of Drinfel'd doubles

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Abstract

Hopf-algebra quantizations of four-dimensional and six-dimensional real classical Drinfel'd doubles are studied by following a direct 'analytic' approach, and the full quantization is explicitly obtained for most of them. Several new four- and six-dimensional quantum algebras are presented and some general features of the method are emphasized.

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1. Introduction

The essential role that Lie bialgebras play in the quantization of Poisson–Lie structures comes from the well-known result by Drinfel'd [1] that establishes a one-to-one correspondence between Poisson–Lie groups and Lie bialgebras. In fact, the concept of classical double [1] is just a reformulation of that of a Lie bialgebra in terms of a (double)-dimensional Lie algebra endowed with a suitable pairing. This 'duplication' process can be iterated by taking into account that the double Lie algebra can be in turn equipped with a quasitriangular Lie bialgebra structure by means of a canonical classical r-matrix.

The Hopf algebra quantization of a double Lie bialgebra is the so-called quantum double, a basic object in quantum group theory (see, for instance, [2–9] and references therein). In particular, quantum doubles are helpful for the explicit construction of quantum R-matrices of quantum groups and supergroups by following a 'universal $\mathcal{T}$-matrix approach' [10] (see also [11–17]). From the physical point of view, σ-models related by Poisson–Lie T-duality are connected with classical doubles (see [18–20] and references therein), while their quantum versions have been considered as symmetries in quantum field theory [21, 22].

We also recall that in [23] the full classification of two- and three-dimensional (3D) real Lie bialgebra structures and their associated Drinfel'd doubles has been obtained. That result is tantamount to the (first order) classification of quantum deformations of 4D and 6D

Drinfel'd doubles. Complementarily, the classification of such doubles has been performed in [20] and [24], respectively, by following a direct approach based on analyticity. Throughout the paper we will preserve the notation and labelling for the Lie bialgebras and Drinfel'd doubles given in [23], although both classifications are fully equivalent (up to the fact that [23] does not consider 6D classical doubles coming from a trivial Lie bialgebra with vanishing cocommutator δ or, equivalently, with Abelian dual $\mathfrak{g}^*$).

The aim of this work is to provide a global insight into the Hopf algebra quantization of 4D and 6D Drinfel'd doubles (DD) by making use of a direct quantization procedure that has been recently used in order to obtain and classify 3D quantum algebras [25]. The motivation for this study is twofold.

Firstly, the family of 6D DD algebras is physically interesting. For instance, it contains the $so(3,1)$ and $so(2,2)$ algebras as well as the (2+1) Poincaré algebra. Therefore the DD quantizations will provide quantum deformations for these algebras. In particular, the class of DDs with $\mathfrak{g} = \mathfrak{r}_3(1)$ (i.e. the Lie algebra generated by two Euclidean translations and one dilation) can be thought as $so(p,q)$ Lie algebras (with $p+q=4$) and some of their contractions [26]. In general, such quasitriangular quantizations turn out to be superpositions of a standard (quasitriangular) quantization plus twists. We will discuss the properties of all these quantum algebras, and we explicitly obtain most of them by making use of a quantization procedure that we introduced in [25].

Secondly, since Lie bialgebras (therefore, DD algebras) provide a classification of first-order quantum deformations, the set of DD quantizations can be considered as an appropriate setting for the search of classification schemes for quantum deformations. In fact, several strong regularities are found within the set of quantum algebras that are presented in this work, and these common facts can be thought of as guidelines for a future research programme of a classification '*à la* Cartan' of quantum groups [25]. Among them, we mention the following common properties of the quantizations:

- All found 4D and 6D DD algebras are non-simple Lie algebras.
- The only functions appearing in the deformed commutation rules and coproducts are exponentials and polynomials [27]. Thus, in each representation, the convergence radius in the complex parameter z is infinity.
- The quantization is obtained by following an 'analytic approach' in the full symmetrized basis of the quantum universal enveloping algebra.
- 'Generalized cocommutativity' (where z is changed of sign) [25] is preserved and the powers of the 'generators' are related with the power of z.
- Every DD deformation presented here is generated by a standard classical r-matrix.

The structure of the paper is as follows: section 2 is devoted to fix the notation. Section 3 presents the study of quantum 4D DD algebras. The following sections develop the quantizations of 6D DD algebras. Finally, some remarks conclude the paper.

2. Drinfel'd double (bi)algebras

Let us consider a Lie bialgebra $(\mathfrak{g}, \delta)$ and a basis $\{x^i\}$ of $\mathfrak{g}$. Such a Lie bialgebra can be characterized by a pair of structure tensors $\left(f_n^{lm}, c_{ij}^k\right)$, i.e.,

$$[x^i, x^j] = f_k^{ij} x^k, \qquad \delta(x^n) = c_{lm}^n x^l \otimes x^m.$$

In this language, the cocycle condition for the cocommutator δ becomes the following compatibility condition between the tensors c and f:

$$f_k^{ab} c_{ij}^k = f_i^{ak} c_{kj}^b + f_i^{kb} c_{kj}^a + f_j^{ak} c_{ik}^b + f_j^{kb} c_{ik}^a. \tag{1}$$

Now we fix a basis $\{X_i\}$ for the dual algebra $\mathfrak{g}^*$ through the following pairing:

$$\langle X_i, X_j\rangle = 0, \qquad \langle x^i, x^j\rangle = 0, \qquad \langle x^i, X_j\rangle = \delta^i_j, \qquad \forall \quad i, j.$$

Then $(\mathfrak{g}^*, \eta)$ is also a Lie bialgebra with structure tensors (f, c), i.e.,

$$[X_i, X_j] = c^k_{ij} X_k, \qquad \eta(X_n) = f^{lm}_n X_l \otimes X_m.$$

This duality leads to the consideration that the pair $(\mathfrak{g}, \mathfrak{g}^*)$ and its associated vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ can be endowed with a Lie algebra structure by means of the commutators,

$$[x^i, x^j] = f^{ij}_k x^k, \qquad [X_i, X_j] = c^k_{ij} X_k, \qquad [x^i, X_j] = c^i_{jk} x^k - f^{ik}_j X_k. \tag{2}$$

This Lie algebra, $D(\mathfrak{g})$, is called the double Lie algebra of $(\mathfrak{g}, \delta)$. Obviously, $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of $D(\mathfrak{g})$, and the compatibility conditions (1) are just Jacobi identities for (2).

Moreover, if $\mathfrak{g}$ is a finite-dimensional Lie algebra, then $D(\mathfrak{g})$ can be endowed with a (quasitriangular) Lie bialgebra structure $(D(\mathfrak{g}), \delta_{\mathrm{DD}})$ generated by the classical r-matrix

$$r = \sum_i x^i \otimes X_i$$

or, equivalently, by its skew-symmetric counterpart

$$\tilde{r} = \frac{1}{2} \sum_i x^i \wedge X_i. \tag{3}$$

In this respect, note that $C = \sum_i (x^i X_i + X_i x^i)$ is always a Casimir operator for the DD algebra. Hence, if we denote $\Omega = \sum_i (x^i \otimes X_i + X_i \otimes x^i)$, then $[1 \otimes Y + Y \otimes 1, \Omega] = 0$ for any generator Y of the DD algebra and $\tilde{r} = r - \frac{1}{2}\Omega$.

The cocommutator δ_{DD} derived from (3) is

$$\delta_{\mathrm{DD}}(x^i) = \delta(x^i) = c^i_{jk} x^j \otimes x^k, \qquad \delta_{\mathrm{DD}}(X_i) = -\eta(X_i) = -f^{jk}_i X_j \otimes X_k.$$

In fact this 'double Lie bialgebra' has as sub-Lie-bialgebras $(\mathfrak{g}, \delta)$ and its dual $(\mathfrak{g}^*, \eta)$.

This paper is based on the integration to all orders in z of the preceding relations, i.e. on the construction of the Δ such that $(\Delta_0(X) = 1 \otimes X + X \otimes 1)$:

$$\Delta = \Delta_0 + z\delta_{\mathrm{DD}} + o[z^2],$$

and, afterwards, of the deformed commutation relations compatible with them.

3. Quantum four-dimensional DD algebras

We shall use the Gomez results and notation for Lie bialgebras [23], that are shown in the tables where for each sub-Lie-bialgebra structure $(\mathfrak{g}, \mathfrak{g}^*)$ and DD algebra all the Lie brackets are given (λ is an essential parameter and $\omega = \pm 1$). In order to simplify the text we do not explicitly write the primitive coproducts Δ_0, and we do not list the undeformed commutation relations that are displayed in the corresponding table. So, for each DD algebra, only coproducts and commutation rules that deformed in the quantization are explicitly presented.

The two 4D DD algebras are the 'standard' one (isomorphic to a semidirect product of the Borel subalgebra $\mathfrak{b}_2$ and R^2, hereafter denoted as $\mathfrak{b}_2 \odot R^2$) and the 'non-standard' one (with $\lambda \neq 0$), which is isomorphic to $gl(2)$. Let us now describe their quantization (see [24] for the connection of both DD algebras with Poisson–Lie T-duality).

Table 1. Four-dimensional DD algebras.

$(\mathfrak{g}, \mathfrak{g}^*)$ DD	$(\mathfrak{b}_2, \mathfrak{b}_2)$ $\mathfrak{b}_2 \odot R^2$	$(\mathfrak{b}_2, \mathfrak{b}_2)$ $gl(2)$
$[x^0, x^1]$	x^1	x^1
$[X_0, X_1]$	X_0	λX_1
$[x^0, X_0]$	x^1	0
$[x^0, X_1]$	$-x^0 - X_1$	$-X_1$
$[x^1, X_0]$	0	λx^1
$[x^1, X_1]$	X_0	$-\lambda x^0 + X_0$

3.1. The case $\mathfrak{b}_2 \odot R^2$

Deformed coproducts integrated from δ_{DD}:

$$\Delta(x^0) = \mathrm{e}^{-zx^1} \otimes x^0 + x^0 \otimes \mathrm{e}^{zx^1}, \qquad \Delta(X_1) = \mathrm{e}^{-zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{zX_0}.$$

Deformed commutation rules compatible with the previous coproduct:

$$[x^0, x^1] = \frac{\sinh(zx^1)}{z}, \qquad [X_0, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^0, X_0] = \frac{\sinh(zx^1)}{z}, \qquad [x^0, X_1] = -\cosh(zX_0)x^0 - \cosh(zx^1)X_1,$$
$$[x^1, X_1] = \frac{\sinh(zX_0)}{z}.$$

To our knowledge, this is a new four-generators quantum algebra, whose additional central element is $x^1 - X_0$. We also remark that, in spite of appearances, also the right-hand side of $[x^0, X_1]$ belongs to the symmetric quantum universal enveloping algebra since

$$-\cosh(zX_0)x^0 - \cosh(zx^1)X_1 = -\mathrm{Sym}\{\cosh(zX_0)x^0\} - \mathrm{Sym}\{\cosh(zx^1)X_1\},$$

where Sym is a linear operator such that

$$\mathrm{Sym}\{A_1 \cdots A_n\} := \frac{1}{n!} \sum_{p \in S_n} p(A_1 \cdots A_n),$$

with S_n the permutation group of n elements (see [25]).

3.2. The $gl(2)$ *case*

The quantization of this DD leads to the deformed coproducts

$$\Delta(x^1) = \mathrm{e}^{z\lambda x^0} \otimes x^1 + x^1 \otimes \mathrm{e}^{-z\lambda x^0}, \qquad \Delta(X_1) = \mathrm{e}^{-zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{zX_0},$$

and to the deformed commutation rule

$$[x^1, X_1] = \frac{\sinh(z(-\lambda x^0 + X_0))}{z}.$$

We mention that $C = \lambda x^0 + X_0$ is a Casimir operator for this quantum algebra, which has been already studied in [28]. Note that the case $\lambda = 0$ is isomorphic to the 'standard' case, as has been pointed out for the DD algebra in [24].

Table 2. DD algebras with $\mathfrak{g} = \mathfrak{r}_3(1)$.

$(\mathfrak{g}, \mathfrak{g}^*)$ DD	$(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	$(\mathfrak{r}_3(1), \mathfrak{so}_3/\mathfrak{sl}_2)$ $\mathfrak{so}(1,3)$	$(\mathfrak{r}_3(1), \mathfrak{sl}_2)'$ $\mathfrak{sl}_2 \odot R^3$	$(\mathfrak{r}_3(1), \mathfrak{s}_3(0))$ $\mathfrak{so}(1,3)$	$(\mathfrak{r}_3(1), \mathfrak{n}_3)$ $\mathfrak{sl}_2 \odot R^3$	$(\mathfrak{r}_3(1), \mathfrak{r}_3(-1)/\mathfrak{s}_3(0))$ $\mathfrak{sl}_2 \odot R^3$
$[x^0, x^1]$	x^1	x^1	x^1	x^1	x^1	x^1
$[x^0, x^2]$	x^2	x^2	x^2	x^2	x^2	x^2
$[x^1, x^2]$	0	0	0	0	0	0
$[X_0, X_1]$	λX_1	λX_2	X_0	λX_2	0	0
$[X_0, X_2]$	$-\lambda X_2$	$-\lambda X_1$	X_1	$-\lambda X_1$	X_1	X_1
$[X_1, X_2]$	X_0	ωX_0	X_2	0	0	ωX_0
$[x^0, X_0]$	0	0	x^1	0	0	0
$[x^0, X_1]$	$x^2 - X_1$	$\omega x^2 - X_1$	$-x^0 - X_1$	$-X_1$	X_1	$\omega x^2 - X_1$
$[x^0, X_2]$	$-x^1 - X_2$	$-\omega x^1 - X_2$	$-X_2$	$-X_2$	$-X_2$	$-\omega x^1 - X_2$
$[x^1, X_0]$	λx^1	$-\lambda x^2$	x^2	$-\lambda x^2$	x^2	x^2
$[x^1, X_1]$	$-\lambda x^0 + X_0$	X_0	X_0	X_0	X_0	X_0
$[x^1, X_2]$	0	λx^0	$-x^0$	λx^0	$-x^0$	$-x^0$
$[x^2, X_0]$	$-\lambda x^2$	λx^1	0	λx^1	0	0
$[x^2, X_1]$	0	$-\lambda x^0$	x^2	$-\lambda x^0$	0	0
$[x^2, X_2]$	$\lambda x^0 + X_0$	X_0	$-x^1 + X_0$	X_0	X_0	X_0

4. Quantum six-dimensional DD algebras: $\mathfrak{g} = \mathfrak{r}_3(1)$

We start with the cases where $\mathfrak{g} = \mathfrak{r}_3(1)$ given in table 2. We stress that the DD's of type $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$, $(\mathfrak{r}_3(1), \mathfrak{so}_3/\mathfrak{sl}_2)$ and $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ are the only DDs for which we have not succeeded in obtaining a complete quantization, since we have not been able to construct the quantum coproduct for the $(\mathfrak{g}, \delta)$ sub-Lie-bialgebra. However, we shall see in the following that it is possible to identify these quantum algebras as Drinfel'd twists of previously known $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{so}(1,3)$ and (2+1) Poincaré algebra deformations. The DD $(\mathfrak{r}_3(1), \mathfrak{sl}_2)'$ differs from $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ in the pairing.

4.1. *Type* $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$

In order to characterize the classical r-matrix $\tilde{r}$ of this DD algebra, we could consider the following pairing-preserving change of basis:

$$\hat{X}_1 = \frac{1}{\sqrt{2}}(X_1 + X_2) + \frac{1}{2\sqrt{2}}(x^1 - x^2), \qquad \hat{X}_2 = \frac{1}{\sqrt{2}}(X_1 - X_2) - \frac{1}{2\sqrt{2}}(x^1 + x^2),$$
$$x'^1 = \frac{1}{\sqrt{2}}(x^1 + x^2), \qquad x'^2 = \frac{1}{\sqrt{2}}(x^1 - x^2).$$

Now, comparing with the commutation relations of the Lie algebra $g_{(\mu_1,\mu_2,\mu_3)}$, a 3-parameter family of graded contractions of $so(2,2)$ [26], we can make the following identification between $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ and the $g_{(\mu_1,\mu_2,\mu_3)}$ generators:

$$N_3 = 2X_0, \qquad J_3 = 2x^0, \qquad J_+ = -x'^2, \qquad N_+ = x'^1, \qquad J_- = -\hat{X}_1, \qquad N_- = \hat{X}_2.$$

The DD algebra $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ is just the $so(2,2)$ algebra $g_{(\lambda,-1/2,\lambda)}$ and its r-matrix is

$$\tilde{r} = r_s + r_{t1} + r_{t2} = \tfrac{1}{2}(x'^1 \wedge \hat{X}_1 + x'^2 \wedge \hat{X}_2) + \tfrac{1}{2}x^0 \wedge X_0 - \tfrac{1}{2}x'^1 \wedge x'^2.$$

Thus, the quantum DD algebra $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ is isomorphic to the standard deformation of $so(2,2)$ generated by r_s plus two non-commuting twists generated by r_{t1} and r_{t2}.

4.2. Types $(\mathfrak{r}_3(1), \mathfrak{so}_3/\mathfrak{sl}_2)$

There are two cases labelled by $\omega\lambda$: when $\omega\lambda > 0$ we have the $(\mathfrak{r}_3(1), \mathfrak{so}_3)$ DD algebra and $\omega\lambda < 0$ corresponds to $(\mathfrak{r}_3(1)\mathfrak{sl}_2)$. Performing the change of basis

$$\hat{X}_1 = X_1 - \frac{\omega}{2}x^2, \qquad \hat{X}_2 = X_2 + \frac{\omega}{2}x^1$$

we obtain that the DD algebra $so(3,1)$ is isomorphic to $g_{(-\lambda,-1/2,\lambda)}$. The r-matrix is

$$\tilde{r} = r_s + r_{t1} + r_{t2} = \tfrac{1}{2}(x^1 \wedge \hat{X}_1 + x^2 \wedge \hat{X}_2) + \tfrac{1}{2}x^0 \wedge X_0 - \tfrac{1}{2}x^1 \wedge x^2.$$

It is the standard deformation of $\mathfrak{so}(1,3)$ [26] plus two non-commuting Reshetikhin twists [29].

4.3. Type $(\mathfrak{r}_3(1), \mathfrak{sl}_2)'$

Through the following change of basis:

$$\hat{X}_1 = X_1 + x^0, \qquad \hat{X}_0 = X_0 - x^1,$$

the Lie algebra turns out to be $g_{(0,-1/2,1)}$ (the (2+1) Poincaré algebra). The classical r-matrix is again the standard one [26] plus a pair of twists:

$$\tilde{r} = r_s + r_{t1} + r_{t2} = \tfrac{1}{2}(x^1 \wedge \hat{X}_1 + x^2 \wedge \hat{X}_2) + \tfrac{1}{2}x^0 \wedge X_0 + \tfrac{1}{2}x^2 \wedge x^0.$$

We recall that previous works on twist deformations of (3+1) Poincaré algebras were restricted to the study of Abelian twists [30].

4.4. Type $(\mathfrak{r}_3(1), \mathfrak{s}_3(0))$

This case is the $\omega \to 0$ limit of the type $(\mathfrak{r}_3(1), \mathfrak{so}_3/\mathfrak{sl}_2)$, and it could be considered as a first step for the quantization of the latter.

Deformed coproducts:

$$\begin{aligned}
\Delta(x^1) &= \cos(z\lambda x^0) \otimes x^1 + \sin(z\lambda x^0) \otimes x^2 + x^1 \otimes \cos(z\lambda x^0) - x^2 \otimes \sin(z\lambda x^0),\\
\Delta(x^2) &= -\sin(z\lambda x^0) \otimes x^1 + \cos(z\lambda x^0) \otimes x^2 + x^1 \otimes \sin(z\lambda x^0) + x^2 \otimes \cos(z\lambda x^0),\\
\Delta(X_1) &= \mathrm{e}^{zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{-zX_0}, \qquad \Delta(X_2) = \mathrm{e}^{zX_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{-zX_0}.
\end{aligned} \tag{4}$$

Deformed commutation rules:

$$\begin{aligned}
[x^1, X_1] &= \frac{\sinh(zX_0)}{z}\cos(z\lambda x^0), \qquad [x^2, X_1] = -\frac{\sin(z\lambda x^0)}{z}\cosh(zX_0),\\
[x^1, X_2] &= \frac{\sin(z\lambda x^0)}{z}\cosh(zX_0), \qquad [x^2, X_2] = \frac{\sinh(zX_0)}{z}\cos(z\lambda x^0).
\end{aligned}$$

The relation with the deformations described in [26] can easily be derived, and this DD algebra $\mathfrak{so}(1,3)$ corresponds to $g_{(-\lambda,-1/2,\lambda)}$ with r-matrix

$$\tilde{r} = r_s + r_t = \tfrac{1}{2}(x^1 \wedge X_1 + x^2 \wedge X_2) + \tfrac{1}{2}x^0 \wedge X_0.$$

We have again the standard deformation of $\mathfrak{so}(1,3)$ plus a Reshetikhin twist r_t.

Note that the canonical basis $\{x^i, X_j\}$ allows a more manageable description since the two Hopf subalgebras with classical commutation rules become apparent (compare with [26]).

4.5. Type $(\mathfrak{r}_3(1), \mathfrak{n}_3)$

This DD algebra corresponds to $g_{(0,-1/2,1)}$ and it is a contraction of both $(\mathfrak{r}_3(1), \mathfrak{sl}_2)$ and $(\mathfrak{r}_3(1), \mathfrak{r}_3(-1)/\mathfrak{s}_3(0))$. Its quantization is simply the $\omega \to 0$ limit of that in section 4.6.

Deformed coproducts:

$$\Delta(x^1) = (1 \otimes x^1 + x^1 \otimes 1) + z(x^2 \otimes x^0 - x^2 \otimes x^0),$$
$$\Delta(X_1) = \mathrm{e}^{zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{-zX_0}, \qquad \Delta(X_2) = \mathrm{e}^{zX_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{-zX_0}.$$

Deformed commutation rules:

$$[x^1, X_1] = \frac{\sinh(zX_0)}{z}, \qquad [x^1, X_2] = -x^0 \cosh zX_0, \qquad [x^2, X_2] = \frac{\sinh(zX_0)}{z}.$$

By using the transformation $X_1' = -\lambda X_1$, $x^{1'} = -x^1/\lambda$, the type $(\mathfrak{r}_3(1), \mathfrak{n}_3)$ is just the limit for $\lambda \to 0$ of the preceding type $(\mathfrak{r}_3(1), \mathfrak{s}_3(0))$. Its r-matrix is

$$\tilde{r} = r_s + r_t = \tfrac{1}{2}(x^1 \wedge X_1 + x^2 \wedge X_2) + \tfrac{1}{2}x^0 \wedge X_0.$$

We have the Poincaré analogue to the previous case (standard deformation [26] plus a twist).

4.6. Types $(\mathfrak{r}_3(1), \mathfrak{r}_3(-1)/\mathfrak{s}_3(0))$

We have $(\mathfrak{r}_3(1), \mathfrak{r}_3(-1))$ for $\omega = +1$, and $(\mathfrak{r}_3(1), \mathfrak{s}_3(0))$ for $\omega = -1$.

Deformed coproducts:

$$\begin{aligned}
\Delta(x^0) &= \cosh(\sqrt{\omega} z x^2) \otimes x^0 + \sqrt{\omega}\sinh(\sqrt{\omega} z x^2) \otimes x^1 \\
&\quad + x^0 \otimes \cosh(\sqrt{\omega} z x^2) - \sqrt{\omega} x^1 \otimes \sinh(\sqrt{\omega} z x^2), \\
\Delta(x^1) &= \frac{\sinh(\sqrt{\omega} z x^2)}{\sqrt{\omega}} \otimes x^0 + \cosh(\sqrt{\omega} z x^2) \otimes x^1 \\
&\quad - x^0 \otimes \frac{\sinh(\sqrt{\omega} z x^2)}{\sqrt{\omega}} + x^1 \otimes \cosh(\sqrt{\omega} z x^2), \\
\Delta(X_1) &= \mathrm{e}^{zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{-zX_0}, \qquad \Delta(X_2) = \mathrm{e}^{zX_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{-zX_0}.
\end{aligned}$$

Deformed commutation rules:

$$[x^0, x^1] = x^1 \cosh[\sqrt{\omega} z x^2], \qquad [x^0, x^2] = \frac{\sinh[\sqrt{\omega} z x^2]}{\sqrt{\omega} z}, \qquad [X_1, X_2] = \omega \frac{\sinh(2zX_0)}{2z},$$
$$[x^0, X_1] = \frac{\sqrt{\omega}}{z} \sinh[\sqrt{\omega} z x^2] \cosh[zX_0] - \cosh[\sqrt{\omega} z x^2] X_1,$$
$$[x^0, X_2] = -\omega x^1 \cosh(zX_0) - X_2 \cosh(\sqrt{\omega} z x^2),$$
$$[x^1, X_0] = \frac{\sinh[\sqrt{\omega} z x^2]}{\sqrt{\omega} z}, \qquad [x^1, X_1] = \cosh[\sqrt{\omega} z x^2] \frac{\sinh[zX_0]}{z},$$
$$[x^1, X_2] = -x^0 \cosh(zX_0), \qquad [x^2, X_2] = \frac{\sinh(zX_0)}{z}.$$

Table 3. DD algebras with $\mathfrak{g} = \mathfrak{r}_3(\rho)$.

$(\mathfrak{g}, \mathfrak{g}^*)$ DD	$(\mathfrak{r}_3(\rho), \mathfrak{n}_3)$ $\mathfrak{r}_3(\rho) \odot R^3$	$(\mathfrak{r}_3(\rho), \mathfrak{r}_3(-\rho))$ $\mathfrak{r}_3(\rho) \odot R^3$	$(\mathfrak{r}_3(\rho), \mathfrak{r}_3(-\rho))'$ $sl_2 \oplus sl_2$
$[x^0, x^1]$	x^1	x^1	x^1
$[x^0, x^2]$	ρx^2	ρx^2	ρx^2
$[x^1, x^2]$	0	0	0
$[X_0, X_1]$	0	X_0	λX_1
$[X_0, X_2]$	0	0	$-\lambda\rho X_2$
$[X_1, X_2]$	$(1+\rho)X_0$	ρX_2	0
$[x^0, X_0]$	0	x^1	0
$[x^0, X_1]$	$(1+\rho)x^2 - X_1$	$-x^0 - X_1$	$-X_1$
$[x^0, X_2]$	$-(1+\rho)x^1 - \rho X_2$	$-\rho X_2$	$-\rho X_2$
$[x^1, X_0]$	0	0	λx^1
$[x^1, X_1]$	X_0	X_0	$-\lambda x^0 + X_0$
$[x^1, X_2]$	0	0	0
$[x^2, X_0]$	0	0	$-\lambda\rho x^2$
$[x^2, X_1]$	0	ρx^2	0
$[x^2, X_2]$	ρX_0	$\rho(-x^1 + X_0)$	$\rho(\lambda x^0 + X_0)$

Note that the use of a fully symmetric basis is implicit since

$$-\omega x^1 \cosh(zX_0) - X_2 \cosh(\sqrt{\omega} z x^2) = -\omega \mathrm{Sym}[x^1 \cosh(zX_0)] - \mathrm{Sym}[X_2 \cosh(\sqrt{\omega} z x^2)].$$

This DD algebra is an iso(2, 1) algebra $g_{(1,-1/2,0)}$ with classical r-matrix

$$\tilde{r} = r_s + r_{t1} + r_{t2} = \frac{1}{2}(x^1 \wedge X_1 + x^2 \wedge X_2) + \frac{1}{2}x^0 \wedge X_0 - \frac{\omega}{2}x^1 \wedge x^2.$$

Once again, we have the standard deformation of the (2+1) Poincaré algebra plus two non-commuting twists, with the same interpretation as for $\mathfrak{so}(1,3)$ of the case in section 4.3.

5. Quantum six-dimensional DD algebras: $\mathfrak{g} = \mathfrak{r}_3(\rho)$

The three cases have been fully quantized by us (table 3).

5.1. Case $(\mathfrak{r}_3(\rho), \mathfrak{n}_3)$

$$\Delta(x^0) = 1 \otimes x^0 + x^0 \otimes 1 - z(1+\rho)(x^1 \otimes x^2 - x^2 \otimes x^1),$$

$$\Delta(X_1) = \mathrm{e}^{zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{-zX_0}, \qquad \Delta(X_2) = \mathrm{e}^{z\rho X_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{-z\rho X_0}.$$

$$[X_1, X_2] = \frac{\sinh(z(1+\rho)X_0)}{z}, \qquad [x^0, X_1] = (1+\rho)x^2 \cosh(zX_0) - X_1,$$

$$[x^0, X_2] = -(1+\rho)x^1 \cosh(z\rho X_0) - \rho X_2, \qquad [x^1, X_1] = \frac{\sinh(zX_0)}{z},$$

$$[x^2, X_2] = \frac{\sinh(z\rho X_0)}{z}.$$

Table 4. DD algebras with $\mathfrak{g} = \mathfrak{r}_3(-1)$ and $\mathfrak{g} = \mathfrak{r}'_3(1)$.

$(\mathfrak{g}, \mathfrak{g}^*)$ DD	$(\mathfrak{r}_3(-1), \mathfrak{n}_3)$ $\mathfrak{r}'_3(1) \odot R^3$	$(\mathfrak{r}_3(-1), \mathfrak{r}'_3(1))$ $\mathfrak{r}'_3(1) \odot R^3$	$(\mathfrak{r}_3(-1), \mathfrak{r}'_3(1))'$ $\mathfrak{sl}_2 \odot R^3$	$(\mathfrak{r}'_3(1), \mathfrak{n}_3)$ $\mathfrak{r}'_3(1) \odot R^3$	$(\mathfrak{r}'_3(1), \mathfrak{n}_3)'$ $sl_2 \odot R^3$
$[x^0, x^1]$	x^1	x^1	x^1	x^1	x^1
$[x^0, x^2]$	$-x^2$	$-x^2$	$-x^2$	$x^1 + x^2$	$x^1 + x^2$
$[x^1, x^2]$	0	0	0	0	0
$[X_0, X_1]$	0	X_0	X_0	0	λX_2
$[X_0, X_2]$	0	0	$-\lambda X_0$	0	0
$[X_1, X_2]$	X_0	$X_0 - X_2$	$X_0 - \lambda X_1 - X_2$	ωX_0	0
$[x^0, X_0]$	0	x^1	$x^1 - \lambda x^2$	0	0
$[x^0, X_1]$	$x^2 - X_1$	$-X_1 - x^0 + x^2$	$-X_1 - x^0 + x^2$	$\omega x^2 - X_1 - X_2$	$-X_1 - X_2$
$[x^0, X_2]$	$-x^1 + X_2$	$-x^1 + X_2$	$\lambda x^0 - x^1 + X_2$	$\omega x^1 - X_2$	$-X_2$
$[x^1, X_0]$	0	0	0	0	0
$[x^1, X_1]$	X_0	X_0	$-\lambda x^2 + X_0$	X_0	X_0
$[x^1, X_2]$	0	0	λx^1	0	0
$[x^2, X_0]$	0	0	0	0	λx^1
$[x^2, X_1]$	0	$-x^2$	$-x^2$	X_0	$X_0 - \lambda x^0$
$[x^2, X_2]$	$-X_0$	$x^1 - X_0$	$x^1 - X_0$	X_0	X_0

5.2. *Case* $(\mathfrak{r}_3(\rho), \mathfrak{r}_3(-\rho))$

$$\Delta(x^0) = e^{zx^1} \otimes x^0 + x^0 \otimes e^{-zx^1}, \qquad \Delta(X_1) = e^{zX_0} \otimes X_1 + X_1 \otimes e^{-zX_0},$$
$$\Delta(x^2) = e^{-z\rho x^1} \otimes x^2 + x^2 \otimes e^{z\rho x^1}, \qquad \Delta(X_2) = e^{z\rho X_0} \otimes X_2 + X_2 \otimes e^{-z\rho X_0}.$$
$$[x^0, x^1] = \frac{\sinh(zx^1)}{z}, \qquad [X_0, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^0, x^2] = \rho x^2 \cosh(zx^1), \qquad [X_1, X_2] = \rho X_2 \cosh(zX_0).$$
$$[x^0, X_0] = \frac{\sinh(zx^1)}{z}, \qquad [x^0, X_1] = -\cosh(zX_0)x^0 - \cosh(zx^1)X_1,$$
$$[x^1, X_1] = \frac{\sinh(zX_0)}{z}, \qquad [x^2, X_1] = \rho x^2 \cosh(zX_0),$$
$$[x^0, X_2] = -\rho X_2 \cosh(zx^1), \qquad [x^2, X_2] = -\frac{\sinh(z\rho(-x^1 + X_0))}{z}.$$

The symmetrization prescription is again preserved despite the non-symmetric shape of some brackets. Note that this DD algebra is self-dual for $\rho = 0$.

5.3. *Case* $(\mathfrak{r}_3(\rho), \mathfrak{r}_3(-\rho))'$

$$\Delta(x^1) = e^{-z\lambda x^0} \otimes x^1 + x^1 \otimes e^{z\lambda x^0}, \qquad \Delta(X_1) = e^{zX_0} \otimes X_1 + X_1 \otimes e^{-zX_0},$$
$$\Delta(x^2) = e^{z\lambda\rho x^0} \otimes x^2 + x^2 \otimes e^{-z\lambda\rho x^0}, \qquad \Delta(X_2) = e^{z\rho X_0} \otimes X_2 + X_2 \otimes e^{-z\rho X_0}.$$
$$[x^1, X_1] = \frac{\sinh(z(-\lambda x^0 + X_0))}{z}, \qquad [x^2, X_2] = \frac{\sinh(z\rho(\lambda x^0 + X_0))}{z}.$$

This DD algebra contains $gl(2)$ as a subalgebra in several different ways and is self-dual for $\rho = 0$.

6. Quantum six-dimensional DD algebras: $\mathfrak{g} = \mathfrak{r}_3(-1)$, $\mathfrak{g} = \mathfrak{r}'_3(1)$

These DD algebras with (table 4) have a semidirect product structure and they have been completely quantized.

6.1. Case $(\mathfrak{r}_3(-1), \mathfrak{n}_3)$

$$\Delta(x^0) = 1 \otimes x^0 + x^0 \otimes 1 - z(x^1 \otimes x^2 - x^2 \otimes x^1),$$
$$\Delta(X_1) = e^{zX_0} \otimes X_1 + X_1 \otimes e^{-zX_0}, \qquad \Delta(X_2) = e^{-zX_0} \otimes X_2 + X_2 \otimes e^{zX_0}.$$
$$[x^0, X_1] = x^2 \cosh(zX_0) - X_1, \qquad [x^1, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^0, X_2] = -x^1 \cosh(zX_0) + X_2, \qquad [x^2, X_2] = -\frac{\sinh(zX_0)}{z}.$$

6.2. Case $(\mathfrak{r}_3(-1), \mathfrak{r}'_3(1))$

$$\Delta(x^0) = e^{zx^1} \otimes x^0 + x^0 \otimes e^{-zx^1} - z\left(x^1 e^{zx^1} \otimes x^2 - x^2 \otimes x^1 e^{-zx^1}\right),$$
$$\Delta(x^2) = e^{zx^1} \otimes x^2 + x^2 \otimes e^{-zx^1},$$
$$\Delta(X_1) = e^{zX_0} \otimes X_1 + X_1 \otimes e^{-zX_0}, \qquad \Delta(X_2) = e^{-zX_0} \otimes X_2 + X_2 \otimes e^{zX_0}.$$
$$[x^0, x^1] = \frac{\sinh(zx^1)}{z}, \qquad [X_0, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^0, x^2] = -x^2 \cosh(zx^1), \qquad [X_1, X_2] = X_0 - X_2 \cosh(zX_0),$$
$$[x^0, X_0] = \frac{\sinh(zx^1)}{z}, \qquad [x^1, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^0, X_1] = -X_1 \cosh(zx^1) - x^0 \cosh(zX_0) + x^2 \cosh(zX_0),$$
$$[x^0, X_2] = -x^1 \cosh(z(x^1 - X_0)) + X_2 \cosh(zx^1),$$
$$[x^2, X_1] = -x^2 \cosh(zX_0), \qquad [x^2, X_2] = \frac{\sinh(z(x^1 - X_0))}{z}.$$

6.3. Case $(\mathfrak{r}_3(-1), \mathfrak{r}'_3(1))'$

In order to get the explicit quantization, we perform a pairing-preserving change of basis:

$$Y_0 = X_0, \qquad Y_1 = \frac{1}{\sqrt{1+\lambda^2}}(\lambda X_1 + X_2), \qquad Y_2 = \frac{1}{\sqrt{1+\lambda^2}}(X_1 - \lambda X_2),$$
$$y^0 = x^0, \qquad y^1 = \frac{1}{\sqrt{1+\lambda^2}}(\lambda x^1 + x^2), \qquad y^2 = \frac{1}{\sqrt{1+\lambda^2}}(x^1 - \lambda x^2).$$

In this new basis, the quantum DD algebra reads

$$\Delta(y^0) = e^{-z\sqrt{1+\lambda^2}y^2} \otimes y^0 + y^0 \otimes e^{z\sqrt{1+\lambda^2}y^2} - z(y^1 \otimes y^2 e^{z\sqrt{1+\lambda^2}y^2} - y^2 e^{-z\sqrt{1+\lambda^2}y^2} \otimes y^1)$$
$$\Delta(y^1) = e^{-z\sqrt{1+\lambda^2}y^2} \otimes y^1 + y^1 \otimes e^{z\sqrt{1+\lambda^2}y^2},$$
$$\Delta(Y_0) = 1 \otimes Y_0 + Y_0 \otimes 1,$$
$$\Delta(Y_1) = \frac{1}{1+\lambda^2}\{(e^{zY_0} + \lambda^2 e^{-zY_0}) \otimes Y_1 + Y_1 \otimes (e^{-zY_0} + \lambda^2 e^{zY_0}) - 2\lambda \sinh(zY_0) \otimes Y_2 + 2\lambda Y_2 \otimes \sinh(zY_0)\},$$
$$\Delta(Y_2) = \frac{1}{1+\lambda^2}\{(e^{-zY_0} + \lambda^2 e^{zY_0}) \otimes Y_2 + Y_2 \otimes (e^{zY_0} + \lambda^2 e^{-zY_0}) - 2\lambda \sinh(zY_0) \otimes Y_1 + 2\lambda Y_1 \otimes \sinh(zY_0)\}.$$
$$[y^0, y^1] = -\frac{1-\lambda^2}{1+\lambda^2} y^1 \cosh(z\sqrt{1+\lambda^2}y^2) + \frac{2\lambda}{1+\lambda^2} \frac{\sinh(2z\sqrt{1+\lambda^2}y^2)}{2z\sqrt{1+\lambda^2}}$$

$$[y^0, y^2] = \frac{2\lambda}{1+\lambda^2} y^1 + \frac{1-\lambda^2}{1+\lambda^2} \frac{\sinh(z\sqrt{1+\lambda^2} y^2)}{z\sqrt{1+\lambda^2}},$$

$$[Y_0, Y_2] = \sqrt{1+\lambda^2} \frac{\sinh(zY_0)}{z}, \qquad [Y_1, Y_2] = -Y_0 + \sqrt{1+\lambda^2} Y_1 \cosh(zY_0),$$

$$[y^0, Y_0] = \frac{\sinh(z\sqrt{1+\lambda^2} y^2)}{z}, \qquad [y^2, Y_1] = \frac{2\lambda}{1+\lambda^2} \frac{\sinh(zY_0)}{z},$$

$$[y^2, Y_2] = \frac{1-\lambda^2}{1+\lambda^2} \frac{\sinh(zY_0)}{z}$$

$$[y^0, Y_1] = \left(\frac{1-\lambda^2}{1+\lambda^2} Y_1 - \frac{2\lambda}{1+\lambda^2} Y_2 \right) \cosh(z\sqrt{1+\lambda^2} y^2)$$
$$- y^2 \cosh(z\sqrt{1+\lambda^2} y^2) + \frac{1-\lambda^2}{1+\lambda^2} y^2 \sinh(z\sqrt{1+\lambda^2} y^2) \sinh(zY_0),$$

$$[y^0, Y_2] = -\sqrt{1+\lambda^2} y^0 \cosh(zY_0) - \left(\frac{2\lambda}{1+\lambda^2} Y_1 - \frac{1-\lambda^2}{1+\lambda^2} Y_2 \right) \cosh(z\sqrt{1+\lambda^2} y^2)$$
$$- \frac{2\lambda}{1+\lambda^2} y^2 \sinh(z\sqrt{1+\lambda^2} y^2) + y^1 \cosh(zY_0),$$

$$[y^1, Y_1] = \frac{\sinh(z\sqrt{1+\lambda^2} y^2)}{z} \cosh(zY_0) - \frac{1-\lambda^2}{1+\lambda^2} \frac{\sinh(zY_0)}{z} \cosh(z\sqrt{1+\lambda^2} y^2),$$

$$[y^1, Y_2] = -\sqrt{1+\lambda^2} y^1 \cosh(zY_0) + \frac{2\lambda}{1+\lambda^2} \frac{\sinh(zY_0)}{z} \cosh(z\sqrt{1+\lambda^2} y^2).$$

6.4. Case $(\mathfrak{r}_3'(1), \mathfrak{n}_3)$

$$\Delta(x^0) = 1 \otimes x^0 + x^0 \otimes 1 + \omega z (x^1 \otimes x^2 - x^2 \otimes x^1),$$
$$\Delta(X_1) = \mathrm{e}^{-zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{zX_0} - z(X_0 \, \mathrm{e}^{-zX_0} \otimes X_2 - X_2 \otimes X_0 \, \mathrm{e}^{zX_0}),$$
$$\Delta(X_2) = \mathrm{e}^{-zX_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{zX_0},$$
$$[X_1, X_2] = \omega \frac{\sinh(2zX_0)}{2z},$$
$$[x^0, X_1] = -X_1 - X_2 + \omega x^2 \cosh(zX_0) - \omega z X_0 x^1 \sinh(zX_0),$$
$$[x^0, X_2] = -X_2 - \omega x^1 \cosh(zX_0), \qquad [x^1, X_1] = \frac{\sinh(zX_0)}{z},$$
$$[x^2, X_1] = X_0 \cosh(zX_0), \qquad [x^2, X_2] = \frac{\sinh(zX_0)}{z}.$$

6.5. Case $(\mathfrak{r}_3'(1), \mathfrak{n}_3)'$

$$\Delta(x^2) = 1 \otimes x^2 + x^2 \otimes 1 + \lambda z (x^0 \otimes x^1 - x^1 \otimes x^0),$$
$$\Delta(X_1) = \mathrm{e}^{-zX_0} \otimes X_1 + X_1 \otimes \mathrm{e}^{zX_0} - z(X_0 \, \mathrm{e}^{-zX_0} \otimes X_2 - X_2 \otimes X_0 \, \mathrm{e}^{zX_0}),$$
$$\Delta(X_2) = \mathrm{e}^{-zX_0} \otimes X_2 + X_2 \otimes \mathrm{e}^{zX_0}.$$
$$[x^1, X_1] = \frac{\sinh(zX_0)}{z}, \qquad [x^2, X_1] = (X_0 - \lambda x^0) \cosh(zX_0), \qquad [x^2, X_2] = \frac{\sinh(zX_0)}{z}.$$

7. Quantum six-dimensional DD algebras: $\mathfrak{g} = \mathfrak{s}_3(\mu)$, $\mathfrak{s}_3(0)$ and $\mathfrak{n}_3$

The four remaining cases of DD algebras are summarized in table 5 and all of them can be fully quantized too.

Table 5. DD algebras with $\mathfrak{g} = \mathfrak{s}_3(\mu)$, $\mathfrak{g} = \mathfrak{s}_3(0)$ and $\mathfrak{g} = \mathfrak{n}_3$.

$(\mathfrak{g}, \mathfrak{g}^*)$ DD	$(\mathfrak{s}_3(\mu), \mathfrak{n}_3)$ $\mathfrak{s}_3(\mu) \odot R^3$	$(\mathfrak{s}_3(\mu), \mathfrak{s}_3(1/\mu))$ $\mathfrak{so}(1,3)$	$(\mathfrak{s}_3(0), \mathfrak{n}_3)$ $\mathfrak{r}_6$	$(\mathfrak{n}_3, \mathfrak{n}_3)$ $\mathfrak{n}_5 \oplus R$
$[x^0, x^1]$	$\mu x^1 - x^2$	$\mu x^1 - x^2$	$-x^2$	0
$[x^0, x^2]$	$x^1 + \mu x^2$	$x^1 + \mu x^2$	x^1	0
$[x^1, x^2]$	0	0	0	x^0
$[X_0, X_1]$	0	$-\lambda X_1/\mu + \lambda X_2$	0	ωX_2
$[X_0, X_2]$	0	$-\lambda X_1 - \lambda X_2/\mu$	0	0
$[X_1, X_2]$	$\mu\omega X_0$	0	ωX_0	0
$[x^0, X_0]$	0	0	0	0
$[x^0, X_1]$	$\mu\omega x^2 - \mu X_1 - X_2$	$-\mu X_1 - X_2$	$\omega x^2 - X_2$	0
$[x^0, X_2]$	$-\mu\omega x^1 + \mu X_1 - \mu X_2$	$X_1 - \mu X_2$	$-\omega x^1 + X_1$	0
$[x^1, X_0]$	0	$-\lambda x^1/\mu - \lambda x^2$	0	$-X_2$
$[x^1, X_1]$	μX_0	$\lambda x^0/\mu + \mu X_0$	0	0
$[x^1, X_2]$	$-X_0$	$\lambda x^0 - X_0$	$-X_0$	0
$[x^2, X_0]$	0	$\lambda x^1 - \lambda x^2/\mu$	0	$\omega x^1 + X_1$
$[x^2, X_1]$	X_0	$-\lambda x^0 + X_0$	X_0	$-\omega x^0$
$[x^2, X_2]$	μX_0	$\lambda x^0/\mu + \mu X_0$	0	0

7.1. *Case* $(\mathfrak{s}_3(\mu), \mathfrak{n}_3)$

$$\Delta(x^0) = 1 \otimes x^0 + x^0 \otimes 1 + \mu\omega z(x^1 \otimes x^2 - x^2 \otimes x^1),$$
$$\Delta(X_1) = \mathrm{e}^{-\mu z X_0}\cos(zX_0) \otimes X_1 + X_1 \otimes \mathrm{e}^{\mu z X_0}\cos(zX_0) - \mathrm{e}^{-\mu z X_0}\sin(zX_0) \otimes X_2 + X_2 \otimes \mathrm{e}^{\mu z X_0}\sin(zX_0),$$
$$\Delta(X_2) = \mathrm{e}^{-\mu z X_0}\cos(zX_0) \otimes X_2 + X_2 \otimes \mathrm{e}^{\mu z X_0}\cos(zX_0) + \mathrm{e}^{-\mu z X_0}\sin(zX_0) \otimes X_1 - X_1 \otimes \mathrm{e}^{\mu z X_0}\sin(zX_0).$$
$$[X_1, X_2] = \frac{\omega}{2}\frac{\sinh(2\mu z X_0)}{z},$$
$$[x^0, X_1] = -\mu X_1 - X_2 - \mu\omega x^1 \sinh(\mu z X_0)\sin(zX_0) + \mu\omega x^2\cosh(\mu z X_0)\cos(zX_0),$$
$$[x^0, X_2] = X_1 - \mu X_2 - \mu\omega x^1\cosh(\mu z X_0)\cos(zX_0) - \mu\omega x^2\sinh(\mu z X_0)\sin(zX_0),$$
$$[x^1, X_1] = \frac{\sinh(\mu z X_0)}{z}\cos(zX_0), \qquad [x^2, X_1] = \cosh(\mu z X_0)\frac{\sin(zX_0)}{z},$$
$$[x^1, X_2] = -\cosh(\mu z X_0)\frac{\sin(zX_0)}{z}, \qquad [x^2, X_2] = \frac{\sinh(\mu z X_0)}{z}\cos(zX_0).$$

7.2. *Case* $(\mathfrak{s}_3(\mu), \mathfrak{s}_3(1/\mu))$

This DD algebra is self-dual for $\mu = 1$ and is isomorphic to $\mathfrak{so}(1,3)$ as an algebra.

$$\Delta(x^1) = \mathrm{e}^{-\frac{\lambda}{\mu}zx^0}\cos(z\lambda x^0) \otimes x^1 + x^1 \otimes \mathrm{e}^{\frac{\lambda}{\mu}zx^0}\cos(z\lambda x^0) - \mathrm{e}^{-\frac{\lambda}{\mu}zx^0}\sin(z\lambda x^0) \otimes x^2 + x^2 \otimes \mathrm{e}^{\frac{\lambda}{\mu}zx^0}\sin(z\lambda x^0),$$
$$\Delta(x^2) = \mathrm{e}^{-\frac{\lambda}{\mu}zx^0}\cos(z\lambda x^0) \otimes x^2 + x^2 \otimes \mathrm{e}^{\frac{\lambda}{\mu}zx^0}\cos(z\lambda x^0) + \mathrm{e}^{-\frac{\lambda}{\mu}zx^0}\sin(z\lambda x^0) \otimes x^1 - x^1 \otimes \mathrm{e}^{\frac{\lambda}{\mu}zx^0}\sin(z\lambda x^0),$$
$$\Delta(X_1) = \mathrm{e}^{-\mu z X_0}\cos(zX_0) \otimes X_1 + X_1 \otimes \mathrm{e}^{\mu z X_0}\cos(zX_0) - \mathrm{e}^{-\mu z X_0}\sin(zX_0) \otimes X_2 + X_2 \otimes \mathrm{e}^{\mu z X_0}\sin(zX_0),$$

$$\Delta(X_2) = e^{-\mu z X_0}\cos(zX_0)\otimes X_2 + X_2\otimes e^{\mu z X_0}\cos(zX_0) + e^{-\mu z X_0}\sin(zX_0)\otimes X_1 - X_1\otimes e^{\mu z X_0}\sin(zX_0).$$

$$[x^1, X_1] = \frac{\sinh\left(z\left(\frac{\lambda}{\mu}x^0 + \mu X_0\right)\right)}{z}\cos(z(\lambda x^0 - X_0)),$$

$$[x^1, X_2] = \frac{\sin(z(\lambda x^0 - X_0))}{z}\cosh\left(z\left(\frac{\lambda}{\mu}x^0 + \mu X_0\right)\right),$$

$$[x^2, X_1] = -\frac{\sin(z(\lambda x^0 - X_0))}{z}\cosh\left(z\left(\frac{\lambda}{\mu}x^0 + \mu X_0\right)\right),$$

$$[x^2, X_2] = \frac{\sinh\left(z\left(\frac{\lambda}{\mu}x^0 + \mu X_0\right)\right)}{z}\cos(z(\lambda x^0 - X_0)).$$

7.3. *Case* $(\mathfrak{s}_3(0), \mathfrak{n}_3)$

$$\Delta(x^0) = 1\otimes x^0 + x^0\otimes 1 + \omega z(x^1\otimes x^2 - x^2\otimes x^1),$$

$$\Delta(X_1) = \cos(zX_0)\otimes X_1 + X_1\otimes\cos(zX_0) - \sin(zX_0)\otimes X_2 + X_2\otimes\sin(zX_0),$$

$$\Delta(X_2) = \cos(zX_0)\otimes X_2 + X_2\otimes\cos(zX_0) + \sin(zX_0)\otimes X_1 - X_1\otimes\sin(zX_0),$$

$$[x^0, X_1] = -X_2 + \omega x^2\cos(zX_0), \qquad [x^2, X_1] = \frac{\sin(zX_0)}{z},$$

$$[x^0, X_2] = X_1 - \omega x^1\cos(zX_0), \qquad [x^1, X_2] = -\frac{\sin(zX_0)}{z}.$$

7.4. *Case* $(\mathfrak{n}_3, \mathfrak{n}_3)$

This is a self-dual DD algebra with classical commutators and deformed coproducts:

$$\Delta(x^2) = 1\otimes x^2 + x^2\otimes 1 + z\omega(x^0\otimes x^1 - x^1\otimes x^0),$$

$$\Delta(X_0) = 1\otimes X_0 + X_0\otimes 1 - z(X_1\otimes X_2 - X_2\otimes X_1).$$

8. Concluding remarks

We have successfully applied the analytical approach described in [25] quantizing Drinfel'd doubles in low dimensions (4D and 6D) and obtained a relevant set of new 6D quantum algebras. The only four cases for which we have not succeeded in the quantization belong to the family of classical doubles containing the Lie algebra A_1, and could be obtained by applying specific twists on the well-known standard quantization of $\mathfrak{so}(1,3)$ and $\mathfrak{sl}_2 \odot R^3$ [26]. In general, it becomes apparent that the complexity of the quantum commutation rules that we have obtained is mainly encoded—by construction—in the deformation of the crossed relations $[x^i, X_j]$ within the DD algebra and that non-simple Lie algebras play a relevant role in quantization.

This study shows that the quantization of the canonical Lie bialgebra structure of a classical double can directly be addressed without making use of the cumbersome construction of the universal $\mathcal{T}$-matrix, which is defined on the canonical dual of the quantum universal enveloping algebra. Work is in progress to extend to the construction of the quantum R-matrix from its classical counterpart r the same procedure that allows us to obtain the coproduct Δ from the cocommutator δ.

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